# Continued fraction expression of the Mathieu series

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**Abstract**

In this paper, we represent a continued fraction expression of Mathieu series by a continued fraction formula of Ramanujan. As application, we obtain some new bounds for Mathieu series.

## 1 Introduction

The infinite series

$$S(r) := \sum_{m=1}^{\infty} \frac{2m}{(m^2+r^2)^2}, \quad (r > 0), \tag{1.1}$$

is called a Mathieu series. It was introduced and studied by Émile Leonard Mathieu in his book [18] devoted to the elasticity of solid bodies. Since its introduction the series $S(r)$ and its various generalizations have attracted many researchers, who established some remarkable properties of these series including the various integral representations, the asymptotic expansions, lower and upper estimates, see e.g. Cerone and Lenard [8], Frontczak [13], Milovanović and Pogány [19], Pogány, Srivastava and Tomovski [21], and references therein.

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An integral representation for the Mathieu series (1.1) is given by Emersleben [12] as

$$S(r) = \frac{1}{r}\int_0^\infty \frac{x}{e^x-1}\sin(rx)dx. \tag{1.2}$$

The integral representation was used by Elbert [11] to derive the asymptotic expansion of $S(r)$:

$$S(r) = \sum_{m=0}^{\infty}(-1)^m \frac{B_{2m}}{r^{2m+2}} = \frac{1}{r^2} - \frac{1}{6r^4} \pm \cdots, \quad (r \to \infty), \tag{1.3}$$

where $B_{2n}$ denote the even indexed Bernoulli numbers defined by the generating function

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi. \tag{1.4}$$

Throughout the paper, we always use notation $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$ be two sequences of real (or complex) numbers with $a_n \neq 0$ for all $n \in \mathbb{N}$. The generalized continued fraction

$$\tau = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}} = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots = b_0 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right) \tag{1.5}$$

is defined as the limit of the $n$th approximant

$$\frac{A_n}{B_n} = b_0 + \mathop{\mathbf{K}}_{k=1}^{n}\left(\frac{a_k}{b_k}\right) \tag{1.6}$$

as $n$ tends to infinity. For the theory of continued fraction, see Cuyt, Petersen, Verdonk, Waadeland and Jones [9] or Lorentzen and Waadeland [16].

Let $r > 0$ and $\operatorname{Re} x > \frac{1}{2}$. Let the continued fraction $CF(r;x)$ with a parameter $r$ be defined by

$$CF(r;x) = \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{1}{4}\left(1+4r^2\right) + \mathbf{K}_{n=1}^{\infty}\left(\frac{\kappa_n}{\left(x-\frac{1}{2}\right)^2+\lambda_n}\right)}, \tag{1.7}$$

where for $n \geq 1$

$$\kappa_n = -\frac{n^4\left(n^2+4r^2\right)}{4(2n-1)(2n+1)}, \quad \lambda_n = \frac{1}{4}(2n^2+2n+1+4r^2). \tag{1.8}$$

The main purpose of this paper is to establish the following continued fraction expression of the Mathieu series.

**Theorem 1.** *Let $r > 0$ and $CF(r;x)$ be defined as (1.7). For all positive integer $k$, we have*

$$S(r) = \sum_{m=1}^{k-1} \frac{2m}{\left(m^2+r^2\right)^2} + CF(r;k), \tag{1.9}$$

*where the sum for $k-1=0$ is stipulated to be zero. In particular,*

$$S(r) = CF(r;1). \tag{1.10}$$

## 2 Some preliminary lemmas

In order to prove Theorem 1, we will prepare some lemmas. The following continued fraction formula of Ramanujan plays an important role in the proof of Theorem 1.

**Lemma 1.** *Let $x, l, m$, and $n$ denote complex numbers. Define*

(2.1)

$$\begin{aligned}P &= P(x,l,m,n)\\ &= \frac{\Gamma\left(\frac{1}{2}(x+l+m+n+1)\right)\Gamma\left(\frac{1}{2}(x+l-m-n+1)\right)\Gamma\left(\frac{1}{2}(x-l+m-n+1)\right)\Gamma\left(\frac{1}{2}(x-l-m+n+1)\right)}{\Gamma\left(\frac{1}{2}(x-l-m-n+1)\right)\Gamma\left(\frac{1}{2}(x-l+m+n+1)\right)\Gamma\left(\frac{1}{2}(x+l-m+n+1)\right)\Gamma\left(\frac{1}{2}(x+l+m-n+1)\right)}.\end{aligned}$$

*Then if either $l, m$, or $n$ is an integer or if* $\operatorname{Re} x > 0$,

$$\frac{1-P}{1+P} = \frac{2lmn}{x^2-l^2-m^2-n^2+1+\mathbf{K}_{j=1}^{\infty}\left(\frac{4(l^2-j^2)(m^2-j^2)(n^2-j^2)}{(2j+1)(x^2-l^2-m^2-n^2+2j^2+2j+1)}\right)}. \tag{2.2}$$

*Proof.* This is Entry 35 of B. C. Berndt [5], p. 157, which was claimed first by Ramanujan [23, 24]. The first published proof was provided by Watson [26]. For the full proof of Entry 35, we refer the reader to L. Lorentzen's paper [14]. □

**Lemma 2.** *The canonical contraction of $b_0 + \mathbf{K}(a_n/b_n)$ with*

$$C_k = A_{2k}, \quad D_k = B_{2k} \quad \text{for } k = 0, 1, 2, \ldots$$

*exists if and only if $b_{2k} \neq 0$ for $k = 0, 1, 2, \ldots$, and is given by*

$$\begin{aligned}b_0 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right) =& b_0 + \frac{a_1 b_2}{a_2 + b_1 b_2} - \frac{a_2 a_3 b_4}{a_3 b_4 + b_2(a_4 + b_3 b_4)} - \frac{a_4 a_5 b_2 b_6}{a_5 b_6 + b_4(a_6 + b_5 b_6)} - \cdots\\ &- \frac{a_{2n} a_{2n+1} b_{2n-2} b_{2n+2}}{a_{2n+1} b_{2n+2} + b_{2n}(a_{2n+2} + b_{2n+1} b_{2n+2})} - \cdots.\end{aligned}$$

*Proof.* It follows from Theorem 12 and Eq. (2.4.3) of L. Lorentzen, H. Waadeland [16], page 83–84. For some applications, interested readers may refer to Berndt [5, p. 121, Eq. (14.2)] or [5, p. 157]. For a canonical contraction of a continued fraction and the related definitions, also see L. Lorentzen, H. Waadeland [16], page 83. □

**Lemma 3.** *$b_0 + \mathbf{K}(a_n/b_n) \approx d_0 + \mathbf{K}(c_n/d_n)$ if and only if there exists a sequence $\{r_n\}$ of complex numbers with $r_0 = 1, r_n \neq 0$ for all $n \in \mathbb{N}$, such that*

$$d_0 = b_0, \quad c_n = r_{n-1} r_n a_n, \quad d_n = r_n b_n \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

*Proof.* See Theorem 9 of L. Lorentzen, H. Waadeland [16], p. 73. □

## 3 The proof of Theorem 1

**Lemma 4.** *Let $r > 0$ and $\operatorname{Re} x > \frac{1}{2}$, then*

$$CF(r;x) - CF(r;x+1) = \frac{2x}{\left(x^2+r^2\right)^2}. \tag{3.1}$$

*Remark* 1. In fact, Lemma 4 was guessed first by the *multiple-correction method* developed in [6, 7]. Mortici [20] made an important contribution in this direction.

*Proof.* By applying Lemma 1 with $(x,l) = (2x-1, 2ri)$ and dividing both sides by $4rmni$, we obtain that for $\operatorname{Re} x > \frac{1}{2}$

$$\frac{1}{4ri}\frac{1-P}{mn(1+P)} = \frac{1}{(2x-1)^2+4r^2-m^2-n^2+1+\mathbf{K}_{j=1}^{\infty}\left(\frac{4(-4r^2-j^2)(m^2-j^2)(n^2-j^2)}{(2j+1)((2x-1)^2+4r^2-m^2-n^2+2j^2+2j+1)}\right)}. \tag{3.2}$$

Now let $m$ tend to zero and $n$ tend to zero, successively. On the right side, we arrive at

$$\frac{1}{(2x-1)^2+1+4r^2+\mathbf{K}_{j=1}^{\infty}\left(\frac{-4j^4(j^2+4r^2)}{(2j+1)((2x-1)^2+2j^2+2j+1+4r^2)}\right)}. \tag{3.3}$$

On the other hand, from the definition of $P$, we see easily that $\lim_{m\to 0} P = 1$. A direct calculation with the use of L'Hospital's rule gives

$$\begin{aligned}\lim_{m\to 0}\frac{1-P}{m(1+P)} &= \lim_{m\to 0}\frac{1}{1+P}\lim_{m\to 0}\frac{1-P}{m} = \frac{1}{2}\lim_{m\to 0}\frac{1-P}{m} = \frac{1}{2}\lim_{m\to 0}\frac{\partial}{\partial m}(1-P) \\ &= \frac{1}{2}\left\{-\psi(-\frac{n}{2}+x-ri)+\psi(\frac{n}{2}+x-ri)+\psi(-\frac{n}{2}+x+ri)-\psi(\frac{n}{2}+x+ri)\right\}.\end{aligned} \tag{3.4}$$

By making use of L'Hospital's rule again, and noting the following classical representation (e.g., see [1, Eq. 6.3.16, p. 259])

$$\psi(z+1) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+z}\right), \quad (z \neq -1,-2,-3,\ldots), \tag{3.5}$$

where $\gamma$ denotes Euler-Mascheroni constant, it follows from (3.4) that

$$\begin{aligned}\lim_{n\to 0}\lim_{m\to 0}\frac{1-P}{mn(1+P)} &= \lim_{n\to 0}\frac{\partial}{\partial n}\left(\lim_{m\to 0}\frac{1}{m}\frac{1-P}{1+P}\right) \\ &= \frac{1}{2}\left(\psi'(x-ri)-\psi'(x+ri)\right) \\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{1}{(x-ri+k)^2}-\sum_{k=0}^{\infty}\frac{1}{(x+ri+k)^2}\right) \\ &= 2ri\sum_{k=0}^{\infty}\frac{x+k}{\left((x+k)^2+r^2\right)^2}.\end{aligned} \tag{3.6}$$

Combining (3.3) and (3.6), we get that for $\operatorname{Re} x > \frac{1}{2}$

$$\tag{3.7} \begin{aligned} \sum_{k=0}^{\infty} \frac{2(x+k)}{((x+k)^2+r^2)^2} =& \frac{4}{(2x-1)^2+1+4r^2+\mathbf{K}_{j=1}^{\infty}\left(\frac{-4j^4(j^2+4r^2)}{(2j+1)((2x-1)^2+2j^2+2j+1+4r^2)}\right)} \\ =& CF(r;x). \end{aligned}$$

Here we used Lemma 3 in the last equality. It is not difficult to check that for $\operatorname{Re} x > \frac{1}{2}$

(3.8)

$$CF(r;x) - CF(r;x+1) = \sum_{k=0}^{\infty} \frac{2(x+k)}{((x+k)^2+r^2)^2} - \sum_{k=0}^{\infty} \frac{2(x+k+1)}{((x+k+1)^2+r^2)^2} = \frac{2x}{(x^2+r^2)^2}.$$

This completes the proof of Lemma 4.

*Proof of Theorem 1.* We use the telescoping method. Theorem 1 follows readily from Lemma 4.

# 4 Some new inequalities for the Mathieu series

The bounds for the Mathieu series attracted many mathematicians like Schröder [25], Emersleben [12], Makai [17] and Diananda [10]. In the past twenty years, many authors like Alzer, Bagdasaryan, Brenner, Guo, Lampret, Milovanović, Mortici, Pogány, Qi, Ruehr, Srivastava, Tomovski, etc. have made important contributions to this research topic, see e.g [2, 3, 15, 19, 20, 21, 22] and references therein. Let us briefly recall some simple results.

Mathieu [18] himself conjectured only the upper bound $S(r) < r^{-2}$, $r > 0$, proved first by Berg [4]. Makai [17] showed the double sided inequalities

$$\tag{4.1} \frac{1}{r^2+\frac{1}{2}} < S(r) < \frac{1}{r^2+\frac{1}{6}}.$$

Alzer *et al.* [2] improved the lower bound to

$$\tag{4.2} \frac{1}{r^2+\frac{1}{2\zeta(3)}} < S(r) < \frac{1}{r^2+\frac{1}{6}},$$

where the constant $1/(2\zeta(3))$ and $1/6$ are sharp.

Milovanović and Pogány [19] stated a composite upper bound of simple structure,

$$\tag{4.3} S(r) \le \begin{cases} \frac{1}{r^2+\frac{1}{4}}, & 0 \le r \le \frac{\sqrt{3}}{2}, \\ \frac{1}{\sqrt{1+4r^2}-1}, & r > \frac{\sqrt{3}}{2}, \end{cases}$$

which is superior to (4.2) in the interval $[0, \sqrt{(5+2\sqrt{3})/6} \approx 1.18772)$.

Let $a_1 = 1$, $b_1 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + r^2$, for $n \geq 1$

$$a_{2n+1} = \frac{n\left(n^2 + 4r^2\right)}{2(2n+1)}, \quad a_{2n} = \frac{n^3}{2(2n-1)}, \tag{4.4}$$

and

$$b_{2n+1} = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{r^2}{2n+1}, \quad b_{2n} = 1. \tag{4.5}$$

By Lemma 2, it is not difficult to prove that

$$CF(r;x) = \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right). \tag{4.6}$$

We let $z = (x - \frac{1}{2})^2 - \frac{1}{4}$, $c_1 = 2$, $d_1 = 2z + 2r^2$,

$$\begin{cases} c_{2n+1} = n(n^2 + 4r^2), \quad c_{2n} = n^3, \\ d_{2n+1} = 2(2n+1)z + 2r^2, \quad d_{2n} = 1. \end{cases} \tag{4.7}$$

It follows easily from Lemma 3 that

**Lemma 5.** *Let* $\mathrm{Re}\, x > \frac{1}{2}$. *With the above notation, we have*

$$CF(r;x) = \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right) = \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{c_n}{d_n}\right). \tag{4.8}$$

**Lemma 6.** *Assume* $x > \frac{1}{2}$. *For all positive integer* $l$, *then*

$$\mathop{\mathbf{K}}_{n=1}^{2l} \left(\frac{a_n}{b_n}\right) < CF(r;x) < \mathop{\mathbf{K}}_{n=1}^{2l-1} \left(\frac{a_n}{b_n}\right). \tag{4.9}$$

*Proof.* As the partial coefficients of the continued fraction $\mathbf{K}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)$ are positive, the assertion is deduced readily from the theory of the continued fraction and the first equality in Lemma 5. □

The following theorem tells us how to obtain sharp bounds for the Mathieu series.

**Theorem 2.** *Let* $r > 0$ *and* $k, l \in \mathbb{N}$. *Let two sequences* $(a(n))_{n\geq 1}, (b(n))_{n\geq 1}$ *be defined by* (4.4) *and* (4.5) *with* $x = k$, *respectively. Then*

$$\sum_{m=1}^{k-1} \frac{2m}{(m^2 + r^2)^2} + \mathop{\mathbf{K}}_{n=1}^{2l} \left(\frac{a_n}{b_n}\right) < S(r) < \sum_{m=1}^{k-1} \frac{2m}{(m^2 + r^2)^2} + \mathop{\mathbf{K}}_{n=1}^{2l-1} \left(\frac{a_n}{b_n}\right). \tag{4.10}$$

*In particular,*

$$\frac{2}{(1+r^2)^2} + \frac{1}{5/2 + r^2} < S(r) < \frac{2}{(1+r^2)^2} + \frac{1}{2 + r^2}, \tag{4.11}$$

$$\frac{2}{(1+r^2)^2} + \frac{4}{(4+r^2)^2} + \frac{1}{13/2 + r^2} < S(r) < \frac{2}{(1+r^2)^2} + \frac{4}{(4+r^2)^2} + \frac{1}{6 + r^2}. \tag{4.12}$$

*Proof.* (4.10) follows readily from Theorem 1 and Lemma 6. Taking $(k, l) = (2, 1)$ and $(k, l) = (3, 1)$ in (4.10), respectively, we can obtain (4.11) and (4.12). $\square$

*Remark* 2. For comparison, our upper bound in (4.11) improves (4.3) when $0 \le r < \sqrt{-2+\sqrt{7}} \approx 0.803587$. It is not hard to check that if

$$r \in \left( \sqrt{\frac{-6+5\zeta(3)}{2+\sqrt{-2+11\zeta(3)-5\zeta^2(3)}}}, \sqrt{\frac{2+\sqrt{-2+11\zeta(3)-5\zeta^2(3)}}{\zeta(3)-1}} \right) \approx (0.0507096, 4.44903),$$

then our lower bound in (4.11) is superior to Alzer's in (4.2). In addition, the bounds in (4.12) are always superior to the bounds in (4.11) for all $r > 0$.

*Remark* 3. Taking $k = 2$ in Theorem 1, letting $r$ tend to zero, and then applying the second equality in Lemma 5, we can deduce the following continued fraction for Apéry number $\zeta(3)$

$$\zeta(3) = 1 + \frac{1}{2^2 \cdot 1} \underset{+}{} \frac{1^3}{1} \underset{+}{} \frac{1^3}{2^2 \cdot 3} \underset{+}{} \frac{2^3}{1} \underset{+}{} \frac{2^3}{2^2 \cdot 5} \underset{+}{} \cdots.$$

Also see Berndt [5], p. 155.

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